Graph Algorithms:
Dynamic Programming

Pedro Barahona

DI/FCT/UNL

Métodos Computacionais

1st Semestre 2019/2020

Dynamic Programming: Algorithms for Graphs

- Most graph properties address optimisation goals, namely

  a. Shortest paths

  b. Minimum Spanning Trees

  c. Minimum Hamiltonian tours (Traveling Salesman)

  d. Minimum number of colours

- Some of these properties (e.g. **a** and **b**, but not **c** nor **d**), can be computed by polynomial algorithms.

- In most cases, algorithms to compute the optima may follow a methodology, **dynamic programming**, based on Mathematical Induction on the Integers:

  - Once an optimal solution is obtained with **n** nodes, extend it to **n+1** nodes.

- We will see two examples of this, in the following algorithms

  - Minimum Spanning Tree – **Prim's Algorithm**

  - Shortest Paths – **Floyd-Warshall's Algorithm**

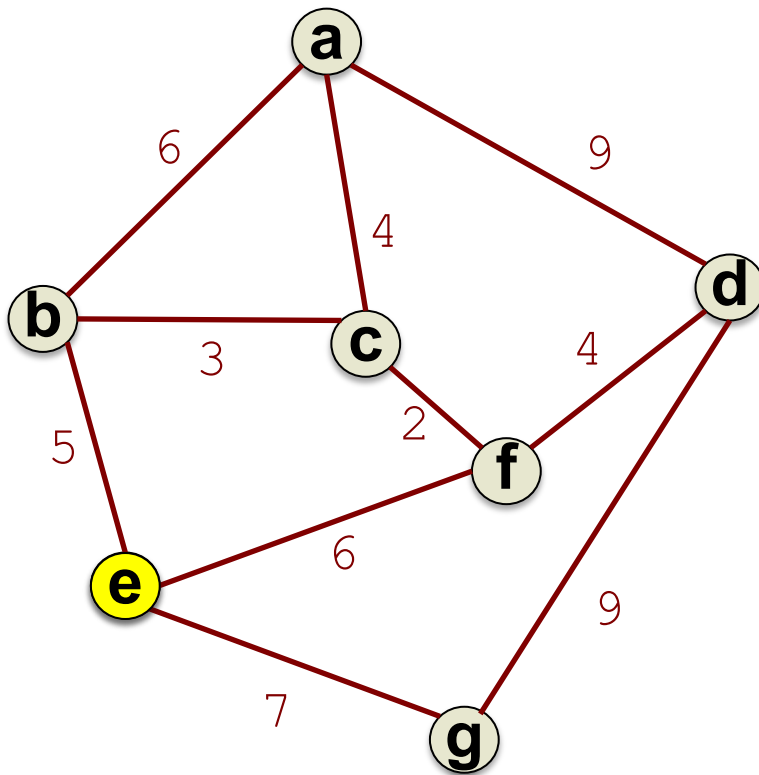# Minimum Spanning Tree: Prim's Algorithm

- A **spanning tree** is a subset of a connected graph that has the topology of a tree and covers all nodes of the graph.

- It has many applications, namely to provide services to a number of sites (the nodes) that can be interconnected in several ways (by a graph), but using the a minimal number of connections that allow all sites to be reached, i.e. a single path connecting any two nodes.

- Among these spanning trees one is usually interested in **minimum spanning trees** (MST) that minimise the sum of the costs of the arcs selected for the tree.

- There are many polynomial algorithms that may be used to compute these MSTs, the most common ones are the Kruskal's and the Prim's algorithms.

- Given the similarities between the latter and the algorithm to check connectedness of a graph, we will address now the **Prim's Algorithm**.

Minimum Spanning Tree: Prim's Algorithm

- The Prim´s algorithm is an example of **Dynamic Programming** that extends a MST with **n** nodes to **n+1** nodes, with an eager selection of the new node (i.e. once the node is selected, the selection **is not backtracked** for alternatives).

- The algorithm can be understood as a process of increasing the size of a current MST, starting with 1 node and ending with all the nodes, and specified as follows:

  - Maintain two sets of nodes: **In** and **Out**, where **In** is the set of nodes already included in a current **MST** and **Out** are those not yet included.

    1. Select arbitrarily a node from the tree to initialise the **In** set, and put the others in the **Out** set;

    2. While there are nodes in the **Out** set,

       i.   Find which node from the **Out** set has an arc of least cost to one connecting it to one of the nodes of the **In** set;

       ii. Transfer the node from the **Out** set to the **In** set and include the least cost arc in the current **MST**.

Minimum Spanning Tree: Prim's Algorithm

- Start with an arbitrary node in the `In` set

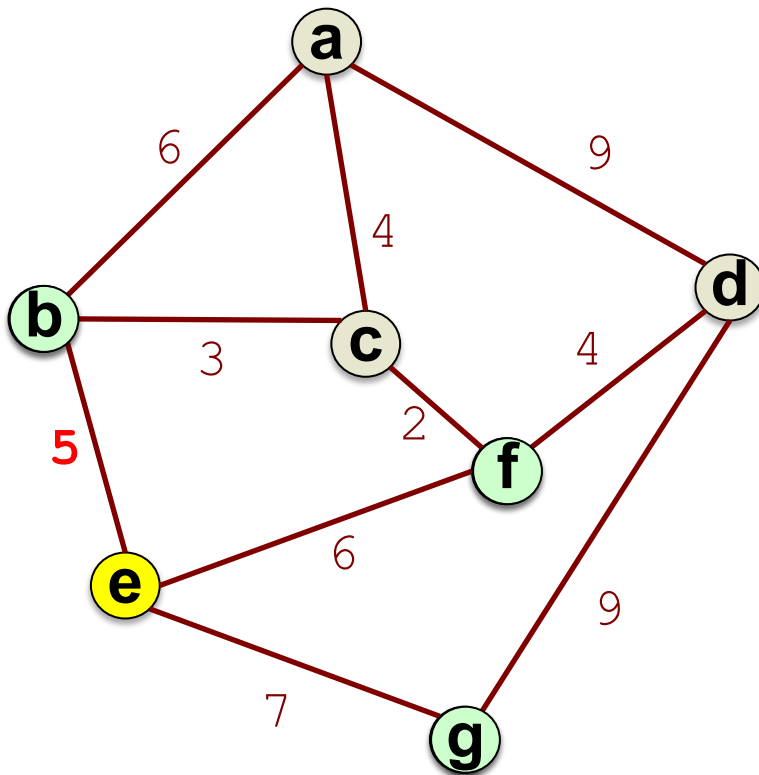- Start with the `Out` set with all the other nodes

- Initialise the `MST` to empty

```
In = [e]
Out = [a,b,c,d,f,g]
MST = {}
```

Minimum Spanning Tree: Prim's Algorithm

- Check all arcs between nodes in the **In** and **Out** sets
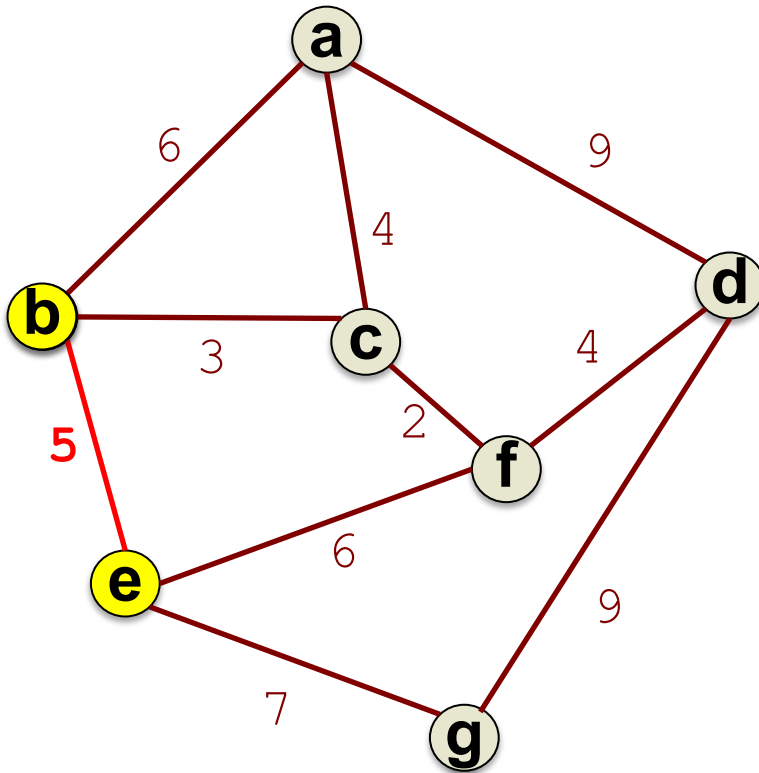
- Chose that with minimum cost


```
In = [e]
Out = [a,b,c,d,f,g]
MST = {}
```

Minimum Spanning Tree: Prim's Algorithm

- Move the out node of the arc from the `Out` to the `In` set.

- Include the arc in the `MST`

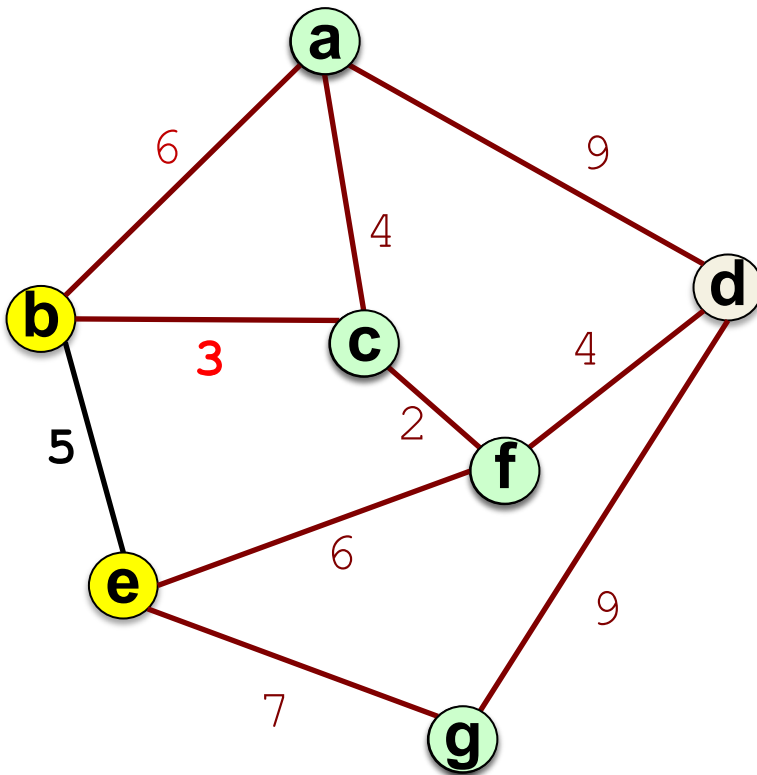```
In = [e,b]
Out = [a,c,d,e,f,g]
MST = {<b,e>}
```

# Minimum Spanning Tree: Prim's Algorithm

- Check all arcs between nodes in the **In** and **Out** sets

- Chose that with minimum cost



```
In = [b,e]
Out = [a,c,d,f,g]
MST = {<b,e>}
```

Minimum Spanning Tree: Prim's Algorithm

- Move the out node of the arc from the `Out` to the `In` set.

- Include the arc in the `MST`

```
In = [b,c,e]
Out = [a,d,f,g]
MST = {<b,e>, <b,c>}
```

Minimum Spanning Tree: Prim's Algorithm

- Check all arcs between nodes in the **In** and **Out** sets

- Chose that with minimum cost


```
In = [b,c,e]
Out = [a,d,f,g]
MST = {<b,e>, <b,c>}
```
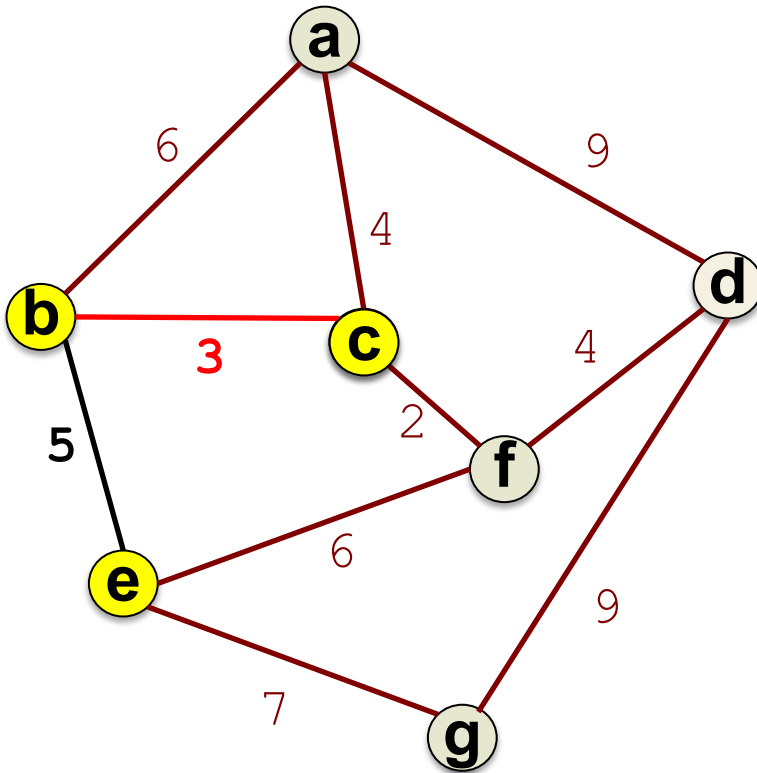
Minimum Spanning Tree: Prim's Algorithm

- Move the out node of the arc from the `Out` to the `In` set.
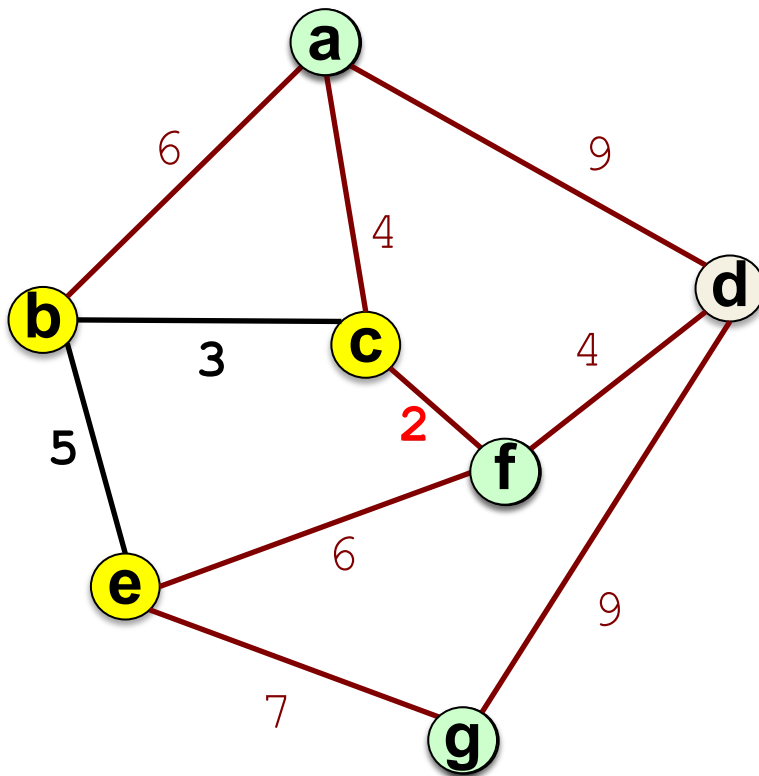
- Include the arc in the `MST`


```
In = [b,c,e,f,]
Out = [a,d,g]
MST = {<b,e>, <b,c>,
        <c,f>}
```

Minimum Spanning Tree: Prim's Algorithm

- Check all arcs between nodes in the `In` and `Out` sets

- Chose that with minimum cost

```
In = [b,c,e,f]
Out = [a,d,g]
MST = {<b,e>, <b,c>,
       <c,f>}
```
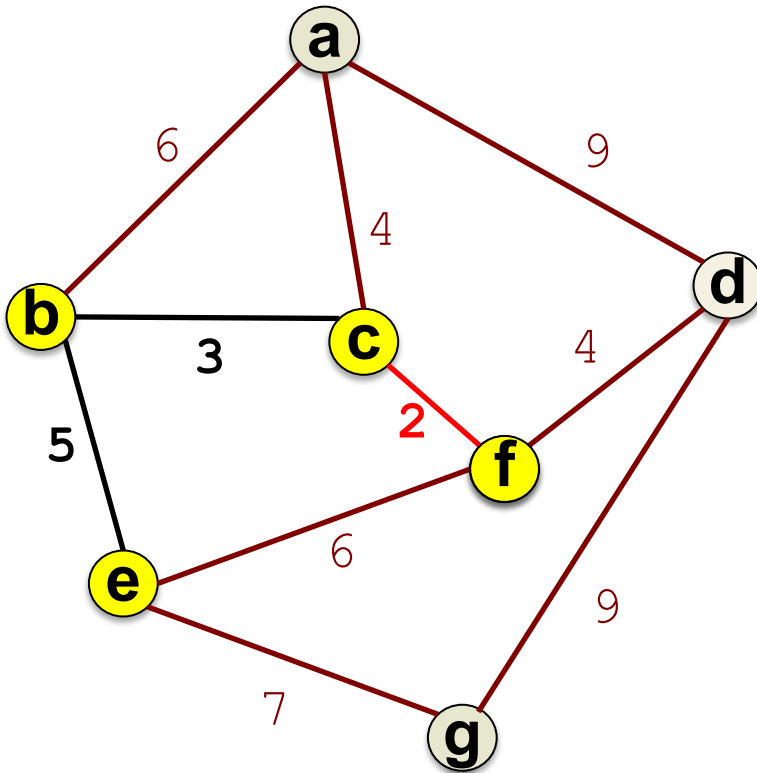
Minimum Spanning Tree: Prim's Algorithm

- Move the out node of the arc from the `Out` to the `In` set.

- Include the arc in the `MST`


```
In = [b,c,d,e,f]
Out = [a,g]
MST = {<b,e>, <b,c>,
        <c,f>, <d,f>}
```

Minimum Spanning Tree: Prim's Algorithm

- Check all arcs between nodes in the `In` and `Out` sets

- Chose that with minimum cost

```
In = [b,c,d,e,f]
Out = [a,g]
MST = {<b,e>, <b,c>,
       <c,f>, <d,f>}
```

Minimum Spanning Tree: Prim's Algorithm

- Move the out node of the arc from the `Out` to the `In` set.
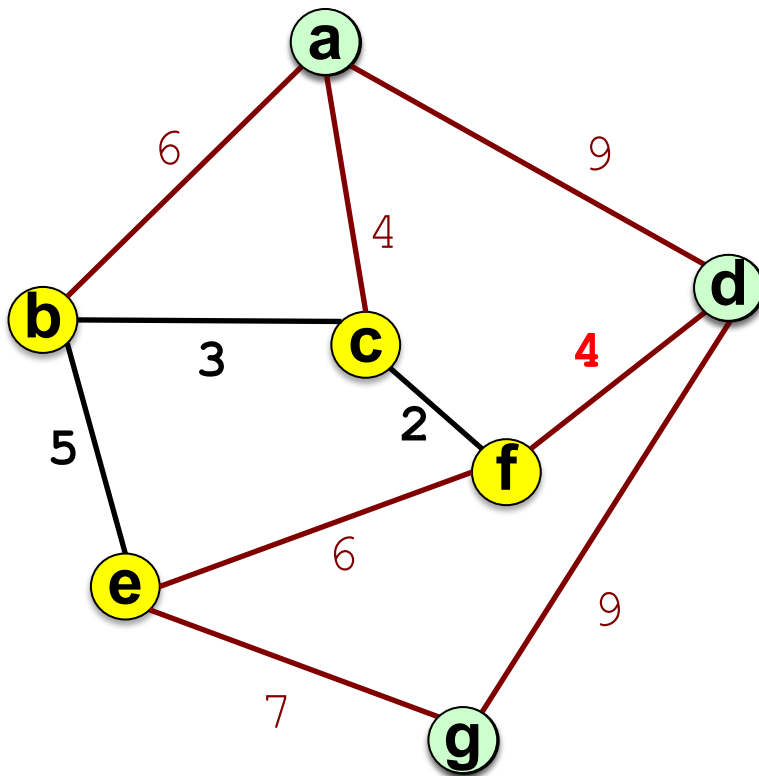
- Include the arc in the `MST`

```
In = [a,b,c,d,e,f]
Out = [g]
MST = {<b,e>, <b,c>,
       <c,f>, <d,f>
       <a,c>}
```
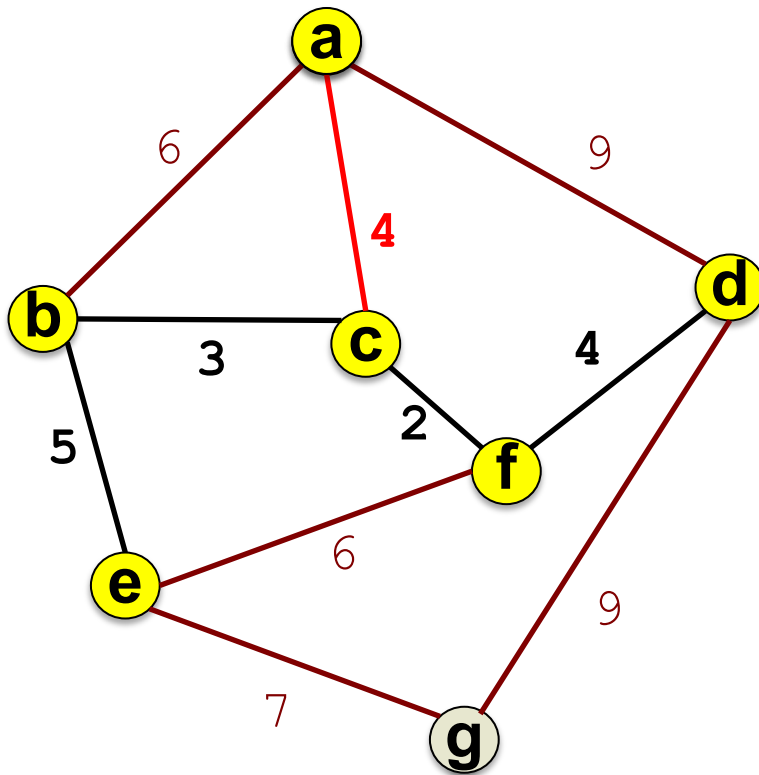
Minimum Spanning Tree: Prim's Algorithm

- Check all arcs between nodes in the `In` and `Out` sets

- Chose that with minimum cost

```
In = [a,b,c,d,e,f]
Out = [g]
MST = {<b,e>, <b,c>,
       <c,f>, <d,f>
       <a,c>}
```
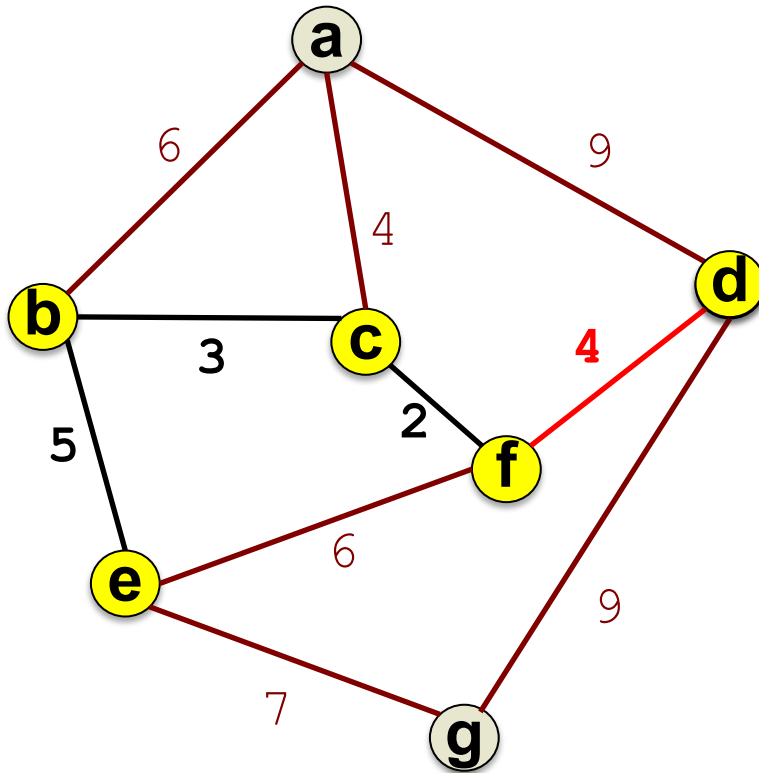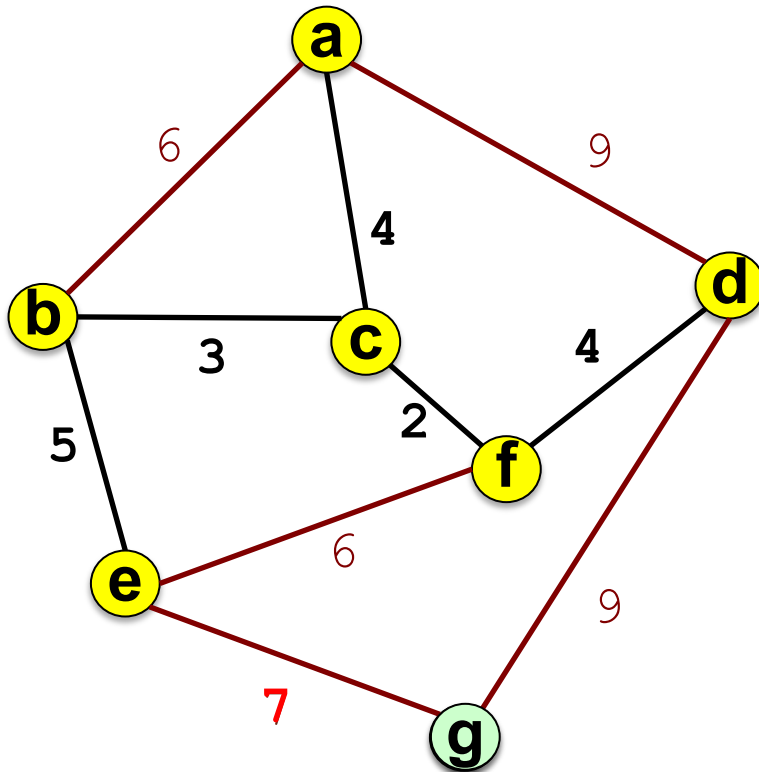
Minimum Spanning Tree: Prim's Algorithm

- Move the out node of the arc from the `Out` to the `In` set.
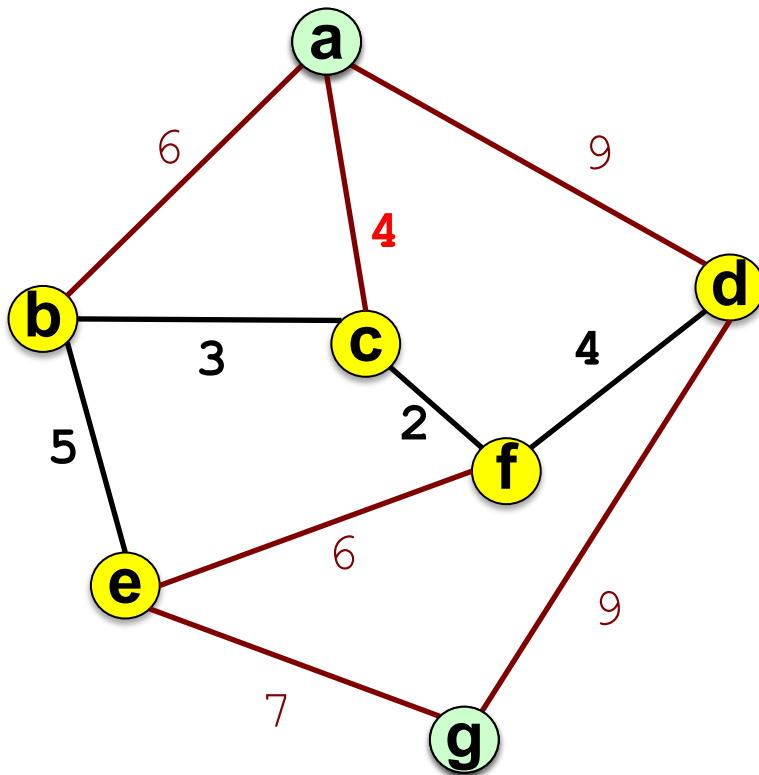
- Include the arc in the `MST`
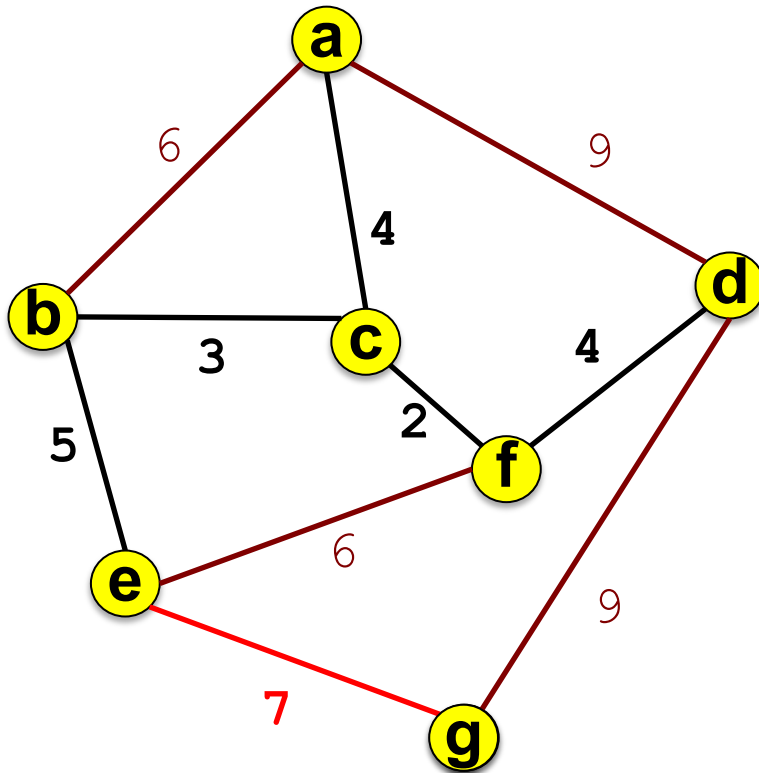

```
In = [a,b,c,d,e,f,g]
Out = []
MST = {<b,e>, <b,c>,
        <c,f>, <d,f>
        <a,c>, <e,g>}
```

Minimum Spanning Tree: Prim's Algorithm

- The `Out` set is now empty

- Return the `MST`.
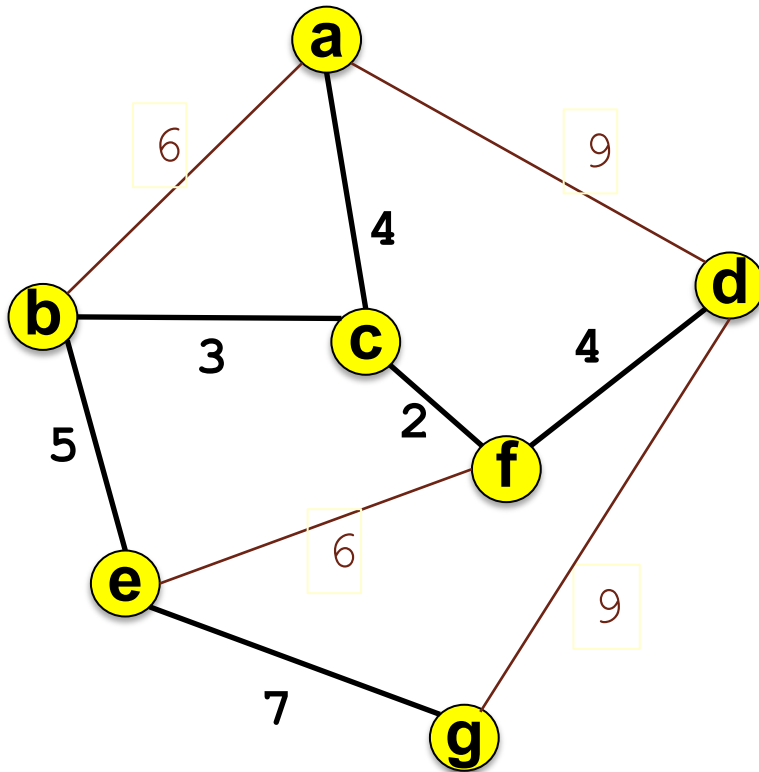
```
In = []
Out = [a,b,c,d,e,f,g]
MST = {<b,e>, <b,c>,
       <a,c>, <c,f>,
       <e,g>, <d,f>}
```

Minimum Spanning Tree: Prim's Algorithm

- Several variants can be used in the implementation of the Prim's algorithm, using appropriate data structures that make it more efficient. Here we present a simple implementation that nonetheless is acceptable for relatively large graphs.

  1. Select arbitrarily a node from the tree to initialise the **In** set (here node 0);
  2. Put the other nodes in the **Out** set;
  3. Initialises the Minimum spanning tree **T** to an empty graph
  4. Update and eventually returns the tree **T** as well as the remaining Out nodes

```python
def prim(G):
    """ Returns the minimum spanning tree of graph G,
    using the prim algorithm"""
    n = len(G)
    In = [0]
    Out = [i for i in range(1,n)]
    row = [-1 for i in range(n)]
    T = [row.copy() for i in range(n)]
    for i in range(n):
        T[i][i] = 0
    while ...:
        ...
    return (T, Out)
```

Minimum Spanning Tree: Prim's Algorithm

- The tree is then updated as follows:

  1. While there are nodes in the **Out** set,
     i. Find the arc of least cost between a node **u** in the **In** set and a node **v** from the **Out** set (if there is one!);
     ii. If no arc is selected, that means the graph is not connected and should be returned (together as the remaining Out nodes)
     iii. Include the least cost arc in the current **MST**.
     iv. Transfer the node from the **Out** set to the **In** set and

```python
while len(Out) > 0:
    min_arc = math.inf
    u = 0
    v = 0
    for i in In:
        for j in Out:
            if G[i][j] > 0 and G[i][j] < min_arc:
                u = i
                v = j
                min_arc = G[i][j]
        ...
```

Minimum Spanning Tree: Prim's Algorithm

- The tree is then updated as follows:

  1. While there are nodes in the **Out** set,

     i. Find the arc of least cost between a node **u** in the **In** set and a node **v** from the **Out** set (if there is one!);

     ii. If no arc is selected, that means the graph is not connected and should be returned (together as the remaining Out nodes); otherwise

     iii. Include the least cost arc in the current **MST**.

     iv. Transfer the node from the **Out** set to the **In** set

```python
while len(Out) > 0:
    ...
            u = i
            v = j
    ...
    if u == 0 and v == 0:
        return (T, Out)
    T[u][v] = G[u][v]
    T[v][u] = G[v][u]
    In.append(v)
    Out.remove(v)
```

Minimum Spanning Tree: Prim's Algorithm

- The complete algorithm is shown below:

```python
def prim(G):
    """ ... """
    n = len(G)
    In = [0]
    Out = [i for i in range(1,n)]
    row = [-1 for i in range(n)]
    T = [row.copy() for i in range(n)]
    for i in range(n):
        T[i][i] = 0
    while len(Out) > 0:
        min_arc = math.inf
        u = 0
        v = 0
        for i in In:
            for j in Out:
                if G[i][j] > 0 and G[i][j] < min_arc:
                    u = i
                    v = j
                    min_arc = G[i][j]
        if u == 0 and v == 0:
            return (T, Out)
        T[u][v] = G[u][v]
        T[v][u] = G[v][u]
        In.append(v)
        Out.remove(v)
    return (T, Out)
```

Dynamic Programming: Graph Algorithms

Minimum Spanning Tree: Prim's Algorithm

- It is easy to prove, by induction, that the algorithm is correct. If T is an MST with least cost with **n** nodes, adding to it the least cost arc will make it an MST with least cost with **n+1** nodes (adding any other arc would lead to a higher cost spanning tree).

- As to the worst cost complexity of the algorithm, with this implementation, we notice that the while loop is executed **n-1** times (n is the number of nodes of the graph, |V|).

```
while len(Out) > 0:
    ...
    for i in In:
        for j in Out:
            ...
```

- Finding the minimal cost arc, when **k** nodes are already in the **In** list, requires two nested loops over ranges with **k** and **n-k** values, i.e. at most **$n^2/4$** (for **k** = n/2) executions of the body of the loop

- All operations in the loop are "basic", and so the complexity of this implementation of the Prim's algorithm is $O(n*n^2/4)$ i.e. **$O(|V|^3)$** (where |V| = n).

- **Note:** Implementations with priority queues and other advanced data structures have better complexity, namely **O(|E|+Vlog|V|).**

Shortest Paths – Floyd-Warshall's Algorithm

- There are many algorithms for finding shortest paths between nodes of weighted graphs. They include algorithms to find one shortest path between two nodes, like the **Dijskstra** algorithm, or to find all shortest paths between any two nodes of the graph, namely the **Floyd-Warshall's** (**FW**) algorithm.

- As the previous one, the **FW** algorithm explores dynamic programming in the following way:

- The initial shortest path between any two nodes, is the direct distance (that can be infinite).

- A list **In** is initialised with all **n** nodes;

- The current shortest distance between two nodes is then updated by checking whether an indirect path exists passing in each of the nodes in list **In**.

- The final result is a matrix with all minimal distances between any two nodes.

Shortest Paths – Floyd-Warshall's Algorithm

- The algorithm can thus be implemented as follows:

   1. Initialise a matrix **S** of shortest paths with the adjacency matrix (that is, only direct distances between any two nodes are initially considered).

      - Of course, nodes that are not directly connected by an arc have a distance of -1 at this stage. For convenience, we will assign them to **inf**.

   1. Now, for all values **k** in the **ln** list (i.e. from 0 to n-1) iterate.

      - In iteration **k**, update **S**, by considering all indirect paths between nodes **i** and **j** passing through node **k.**

   3. After the last iteration, matrix S contains all shortest paths between any two nodes of G.

- Notice that this algorithm only computes the paths with shortest distance between any two nodes but does not return what these paths are.

   - In fact, a small addition to the algorithm, coming shortly, allows the paths to be obtained.

Shortest Paths – Floyd-Warshall's Algorithm

- The external for loop guarantees that all paths, between nodes i and j, consider, all the paths through nodes k (1, 2, 3, …, n), previously computed.

- The shortest paths are updated by considering the triangular inequality, with paths passing through the previous values of **k**.

```python
def floyd(G):
 """ Returns the minimum distances between any two modes
     of graph G, using the Floyd-Warshall's algorithm."""
    n = len(G)
    S = [ [ math.inf for i in range(n) ] for j in range(n) ]
    for i in range(n):
        for j in range(n):
            if G[i][j] != -1:
                S[i][j]= G[i][j]
    In = [i for i in range(n)]
    for k in In:
      for i in range(n):
        for j in range(n):
            if S[i][k] + S[k][i]  < S[i][j]:
                S[i][j]  = S[i][k]  + S[k][j]
    return S
```

Shortest Paths – Floyd-Warshall's Algorithm

- The correction of the algorithm can be proved by induction on the number of nodes considered in indirect paths (left as exercise).

- As to the complexity, it is easy to see that the algorithm requires 3 nested loops of size **n**, with a basic operation in the body,

```
for k in In:     # In = [0..n-1]
    for i in range(n):
        for j in range(n):
```

- The complexity of the algorithm is thus **O(|V|³).**

- Notice that algorithms to compute shortest paths between 2 nodes, like the Dijkstra algorithm, have complexity **O(|V|²)**, but they do not consider te distance between all the nodes.

Path Reconstruction – Floyd-Warshall's Algorithm

- The previous algorithm does not provide the shortest paths between any two nodes, but rather the shortest distances of any path between the nodes.

- Nevertheless, these paths may be easily reconstructed if a matrix is computed during the FW algorithm, to it possible to later compute the shortest path from some node **i** to another node **j**.

- Matrix **Next** plays this role. In such matrix, `Next[i][j] = k` means that to follow the shortest path to node **j**, the path should start by the arc `i` ➞ k.

- All that is needed is to compute matrix **Next** during the FW algorithm. To do so, all values `Next[i][j]` should be initialised with **j** (in the beginning, i.e. before exploring the graph, the best path is the direct path).

- In fact, if there is no connection between nodes i and j, `Next[i][j]` should be initialised to `inf`, to account for that non-connection.

- Then, if a better path is found through node **k**, `Next[i][j]` must be updated to `Next[i][j]`, i.e. to go from **i** to **j**, one should start in the best arc to go from i to k.

- Better paths are found in the inner loop of the FW algorithm, so one needs simply to add some extra instructions to function floyd just developed.

Path Reconstruction – Floyd-Warshall's Algorithm

- The initialisation of matrix Next (i.e. **Next[i][j] = j**) can be implemented as:

```
Next = [ [ j for j in range(n) ] for i in range(n) ]
```

- If there is no connection, this should be accounted for in **Next**

```
        if G[i][j] != -1:
            ...
        else:
            Next[i][j] = math.inf
```

- The update of the elements of Next may be done in the inner loop of the floyd function, taking into account that

  - If a better path is found, through node **k**, **Next[i][j]** must be updated to **Next[i][j]**, i.e. to go from **i** to **j**, one should start in the best arc to go from **i** to **k**.

```
        for k in In:
            for i in range(n):
                for j in range(n):
                    if S[i][k] + S[k][i]  < S[i][j]:
                        S[i][j]  = S[i][k]  + S[k][j]
                        Next[i][j] = Next[i][k]
```

Path Reconstruction – Floyd-Warshall's Algorithm

- The completed Floyd function is thus shown below (changes in bold).

```python
def floyd(G):
    """ ... """
    n = len(G)
    S = [ [ math.inf for i in range(n) ] for j in range(n) ]
    Next = [ [ j for j in range(n) ] for i in range(n) ]
    for i in range(n):
        for j in range(n):
            if G[i][j] != -1:
                S[i][j]= G[i][j]
            else:
                Next[i][j] = math.inf
    In = [i for i in range(n)]
    for k in In:
      for i in range(n):
        for j in range(n):
            if S[i][k] + S[k][i]  < S[i][j]:
                S[i][j]  = S[i][k]  + S[k][j]
                Next[i][j] = Next[i][k]
    return (S, Next)
```

- Once the matrix Next is returned the path between any two nodes, u and v, can be obtained by following the trail indicated by this matrix, as shown below

```python
def path(u,v,Next):
    """ Returns the shortest path between nodes u and v,
    according to matrix Next computed with function floyd."""
    if Next[u][v] == math.inf:
        return []
    P = [u]
    while u != v:
        u = Next[u][v]
        P.append(u)
    return P
```

- The first test checks whether there is any path between nodes **u** and **v**. If not it returns an empty path.

- Otherwise the path is "reconstructed", starting in node **u** …
- … and continuing until reaching node v

- With this reconstruction technique, the complexity of the FW algorithm is not changed, and the paths are only computed when needed.

Dynamic Programming: Graph Algorithms